

HYPERELLIPTIC JACOBIANS AND PROJECTIVE LINEAR GALOIS GROUPS

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1. INTRODUCTION

In [14] the author proved that in characteristic 0 the jacobian $J(C) = J(C_f)$ of a hyperelliptic curve

$$C = C_f : y^2 = f(x)$$

has only trivial endomorphisms over an algebraic closure K_a of the ground field K if the Galois group $\text{Gal}(f)$ of the irreducible polynomial $f \in K[x]$ is “very big”. Namely, if $n = \deg(f) \geq 5$ and $\text{Gal}(f)$ is either the symmetric group \mathbf{S}_n or the alternating group \mathbf{A}_n then the ring $\text{End}(J(C_f))$ of K_a -endomorphisms of $J(C_f)$ coincides with \mathbf{Z} . Later the author [15] proved that $\text{End}(J(C_f)) = \mathbf{Z}$ for an infinite series of $\text{Gal}(f) = \mathbf{L}_2(2^r) := \text{PSL}_2(\mathbf{F}_{2^r})$ and $n = 2^r + 1$ (with $r \geq 3$ and $\dim(J(C_f)) = 2^{r-1}$) or when $\text{Gal}(f)$ is the Suzuki group $\mathbf{Sz}(2^{2r+1})$ and $n = 2^{2(2r+1)} + 1$ (with $\dim(J(C_f)) = 2^{4r+1}$). He also proved the same assertion when $n = 11$ or 12 and $\text{Gal}(f)$ is the Mathieu group \mathbf{M}_{11} or \mathbf{M}_{12} . (In those cases $J(C_f)$ has dimension 5.)

We refer the reader to [12], [13], [7], [8], [9], [14], [15] for a discussion of known results about, and examples of, hyperelliptic jacobians without complex multiplication.

In the present paper we prove that $\text{End}(J(C_f)) = \mathbf{Z}$ when the set $\mathfrak{R} = \mathfrak{R}_f$ of roots of f can be identified with the $(m-1)$ -dimensional projective space $\mathbf{P}^{m-1}(\mathbf{F}_q)$ over a finite field \mathbf{F}_q of odd characteristic in such a way that $\text{Gal}(f)$, viewed as a permutation group of \mathfrak{R}_f , becomes either the projective linear group $\text{PGL}(m, \mathbf{F}_q)$ or the projective special linear group $\mathbf{L}_m(q) := \text{PSL}(m, \mathbf{F}_q)$. Here we assume that $m > 2$. In this case

$$n = \deg(f) = \#(\mathbf{P}^{m-1}(\mathbf{F}_q)) = \frac{q^m - 1}{q - 1}$$

and $\dim(J(C_f))$ is $[(\frac{q^m-1}{q-1} - 1)/2]$, i.e. the integral part of $(\frac{q^m-1}{q-1} - 1)/2$.

Our proof is based on a result of Guralnick [3], who proved that in the “generic” case the dimension of each nontrivial irreducible representation of $\mathbf{L}_m(q)$ in characteristic 2 is greater than or equal to

$$2[(\frac{q^m-1}{q-1} - 1)/2].$$

We also discuss the similar problem when K has prime characteristic > 2 . It turns out that $\text{End}(J(C_f)) = \mathbf{Z}$ under an additional assumption that m is even (i.e., when n is even). The case of $n = 12$ and $\text{Gal}(f) = \mathbf{M}_{12}$ is also treated.

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2. MAIN RESULTS

Throughout this paper we assume that K is a field with $\text{char}(K) \neq 2$. We fix its algebraic closure K_a and write $\text{Gal}(K)$ for the absolute Galois group $\text{Aut}(K_a/K)$. If X is an abelian variety defined over K then we write $\text{End}(X)$ for the ring of K_a -endomorphisms of X .

Suppose $f(x) \in K[x]$ is a separable polynomial of degree $n \geq 5$. Let $\mathfrak{R} = \mathfrak{R}_f \subset K_a$ be the set of roots of f , let $K(\mathfrak{R}) = K(\mathfrak{R})$ be the splitting field of f and $\text{Gal}(f) := \text{Gal}(K(\mathfrak{R})/K)$ the Galois group of f , viewed as a subgroup of $\text{Perm}(\mathfrak{R})$. Let C_f be the hyperelliptic curve $y^2 = f(x)$. Let $J(C_f)$ be its jacobian, $\text{End}(J(C_f))$ the ring of K_a -endomorphisms of $J(C_f)$.

Theorem 2.1. *Assume that there exist a positive integer $m > 2$ and an odd power prime q such that $n = \frac{q^m - 1}{q - 1}$ and $\text{Gal}(f)$ contains a subgroup isomorphic to $\mathbf{L}_m(q)$. (E.g., $\text{Gal}(f)$ is isomorphic either to $\text{PGL}_m(\mathbf{F}_q)$ or to $\mathbf{L}_m(q) = \text{PSL}_m(\mathbf{F}_q)$.)*

Then either $\text{End}(J(C_f)) = \mathbf{Z}$ or m is odd, $\text{char}(K) > 0$ and $J(C_f)$ is a supersingular abelian variety.

Remark 2.2. Clearly m is even if and only if n is even.

Remark 2.3. Replacing K by $K(\mathfrak{R})^{\mathbf{L}_m(q)}$, we may assume that

$$\text{Gal}(f) = \mathbf{L}_m(q).$$

Also, taking into account that $\mathbf{L}_m(q)$ is simple non-abelian and replacing K by its abelian extension obtained by adjoining all 2-power roots of unity, we may assume that K contains all 2-power roots of unity.

Theorem 2.4. *Suppose $n = 12$ and $\text{Gal}(f)$ is isomorphic to the Mathieu group M_{12} . Then $\text{End}(J(C_f)) = \mathbf{Z}$.*

Remark 2.5. When $\text{char}(K) = 0$ the assertion of Theorem 2.4 is proven in [15]. Taking into account that M_{12} is simple non-abelian and replacing K by its abelian extension obtained by adjoining all 2-power roots of unity, we may assume that K contains all 2-power roots of unity.

We will prove Theorems 2.1 and 2.4 in §4.

3. PERMUTATION GROUPS AND PERMUTATION MODULES

Let B be a finite set consisting of $n \geq 5$ elements. We write $\text{Perm}(B)$ for the group of permutations of B . A choice of ordering on B gives rise to an isomorphism

$$\text{Perm}(B) \cong \mathbf{S}_n.$$

Let G be a subgroup of $\text{Perm}(B)$. For each $b \in B$ we write G_b for the stabilizer of b in G ; it is a subgroup of G .

Remark 3.1. Assume that the action of G on B is transitive. It is well-known that each G_b is of index n in G and all the G_b 's are conjugate in G . Each conjugate of G_b in G is the stabilizer of a point of B . In addition, one may identify the G -set B with the set of cosets G/G_b with the standard action by G .

Let \mathbf{F} be a field. We write \mathbf{F}^B for the n -dimensional \mathbf{F} -vector space of maps $h : B \rightarrow \mathbf{F}$. The space \mathbf{F}^B is provided with a natural action of $\text{Perm}(B)$ defined as follows. Each $s \in \text{Perm}(B)$ sends a map $h : B \rightarrow \mathbf{F}$ into $sh : b \mapsto h(s^{-1}(b))$. The permutation module \mathbf{F}^B contains the $\text{Perm}(B)$ -stable hyperplane

$$(\mathbf{F}^B)^0 = \{h : B \rightarrow \mathbf{F} \mid \sum_{b \in B} h(b) = 0\}$$

and the $\text{Perm}(B)$ -invariant line $\mathbf{F} \cdot 1_B$ where 1_B is the constant function 1. The quotient $\mathbf{F}^B/(\mathbf{F}^B)^0$ is a trivial 1-dimensional $\text{Perm}(B)$ -module.

Clearly, $(\mathbf{F}^B)^0$ contains $\mathbf{F} \cdot 1_B$ if and only if $\text{char}(\mathbf{F})$ divides n . If this is *not* the case then there is a $\text{Perm}(B)$ -invariant splitting

$$\mathbf{F}^B = (\mathbf{F}^B)^0 \oplus \mathbf{F} \cdot 1_B.$$

Clearly, \mathbf{F}^B and $(\mathbf{F}^B)^0$ carry natural structures of G -modules. Their characters depend only on the characteristic of \mathbf{F} .

Let us consider the case of $\mathbf{F} = \mathbf{Q}$. Then the character of \mathbf{Q}^B is called the *permutation character* of B . Let us denote by $\chi = \chi_B : G \rightarrow \mathbf{Q}$ the character of $(\mathbf{Q}^B)^0$. Clearly, $1 + \chi$ is the permutation character of B .

Now, let us consider the case of $\mathbf{F} = \mathbf{F}_2$. If n is even then let us define the $\text{Perm}(B)$ -module

$$Q_B := (\mathbf{F}_2^B)^0 / (\mathbf{F}_2 \cdot 1_B).$$

If n is odd then let us put

$$Q_B := (\mathbf{F}_2^B)^0.$$

Remark 3.2. Clearly, Q_B is a faithful G -module. If n is odd then $\dim_{\mathbf{F}_2}(Q_B) = n - 1$. If n is even then $\dim_{\mathbf{F}_2}(Q_B) = n - 2$.

Let $G^{(2)}$ be the set of 2-regular elements of G . Clearly, the Brauer character of the G -module \mathbf{F}_2^B coincides with the restriction of $1 + \chi_B$ to $G^{(2)}$. This implies easily that the Brauer character of the G -module $(\mathbf{F}_2^B)^0$ coincides with the restriction of χ_B to $G^{(2)}$.

Remark 3.3. Let us denote by $\phi_B = \phi$ the Brauer character of the G -module Q_B . One may easily check that ϕ_B coincides with the restriction of χ_B to $G^{(2)}$ if n is odd and with the restriction of $\chi_B - 1$ to $G^{(2)}$ if n is even.

We refer to [15] for a discussion of the following definition.

Definition 3.4. Let V be a vector space over a field \mathbf{F} , let G be a group and $\rho : G \rightarrow \text{Aut}_{\mathbf{F}}(V)$ a linear representation of G in V . We say that the G -module V is very *simple* if it enjoys the following property:

If $R \subset \text{End}_{\mathbf{F}}(V)$ is an \mathbf{F} -subalgebra containing the identity operator Id such that

$$\rho(\sigma)R\rho(\sigma)^{-1} \subset R \quad \forall \sigma \in G$$

then either $R = \mathbf{F} \cdot \text{Id}$ or $R = \text{End}_{\mathbf{F}}(V)$.

Remarks 3.5. (i) If G' is a subgroup of G and the G' -module V is very simple then obviously the G -module V is also very simple.

(ii) A very simple module is absolutely simple (see [15], Remark 2.2(ii)).

(iii) If $\dim_{\mathbf{F}}(V) = 1$ then obviously the G -module V is very simple.

- (iv) Assume that the G -module V is very simple and $\dim_{\mathbf{F}}(V) > 1$. Then V is not induced from a subgroup G (except G itself) and is not isomorphic to a tensor product of two G -modules, whose \mathbf{F} -dimension is strictly less than $\dim_{\mathbf{F}}(V)$ (see [15], Examples 7.1).
- (v) If $\mathbf{F} = \mathbf{F}_2$ and G is *perfect* then properties (ii)-(iv) characterize the very simple G -modules (see [15], Th. 7.7).

The following statement provides a criterion of very simplicity over \mathbf{F}_2 .

Theorem 3.6. *Suppose a positive integer $N > 1$ and a group H enjoy the following properties:*

- H does not contain a subgroup of index dividing N except H itself.
- Let $N = ab$ be a factorization of N into a product of two positive integers $a > 1$ and $b > 1$. Then either there does not exist an absolutely simple $\mathbf{F}_2[H]$ -module of \mathbf{F}_2 -dimension a or there does not exist an absolutely simple $\mathbf{F}_2[H]$ -module of \mathbf{F}_2 -dimension b .

Then each absolutely simple $\mathbf{F}_2[H]$ -module of \mathbf{F}_2 -dimension N is very simple.

Proof. This is Corollary 4.12 of [15]. □

Theorem 3.7. *Suppose that there exist a positive integer $m > 2$ and an odd power prime q such that $n = \frac{q^m - 1}{q - 1}$. Suppose G is a subgroup of \mathbf{S}_n . Suppose G contains a subgroup isomorphic to $\mathbf{L}_m(q)$. Then the G -module Q_B is very simple.*

The rest of this section is devoted to the proof of Theorem 3.7. In light of Remark 3.5(ii), we may assume that $G = \mathbf{L}_m(q)$.

“Generic” case. Assume that $(m, q) \neq (4, 3)$. Then it follows from Theorem 1.1 (applied to $p = 2$) and the Table III of [3] that each nontrivial (absolutely) irreducible representation of $\mathbf{L}_m(q)$ in characteristic 2 has dimension which is greater or equal than $N := \dim_{\mathbf{F}_2}(Q_B)$. Taking into account that $\mathbf{L}_m(q)$ is (simple) not solvable and Q_B is a faithful $\mathbf{L}_m(q)$ -module, we conclude that Q_B is absolutely simple.

Now we claim that the group $G = \mathbf{L}_m(q)$ does not contain a subgroup of index dividing $N := \dim_{\mathbf{F}_2}(Q_B)$ except G itself.

Indeed, if G' is a subgroup of G such that $G' \neq G$ and $[G : G']$ divides $\dim_{\mathbf{F}_2}(Q_B)$ then the simple group G acts faithfully on $B' = G/G'$ and therefore $[G : G'] \geq 5$. In particular, we get a faithful G -module $Q_{B'}$, whose dimension is strictly less than $\dim_{\mathbf{F}_2}(Q_B)$.

Since each strict divisor a of N lies strictly between 1 and N , there does not exist an absolutely simple $\mathbf{F}_2[G]$ -module of \mathbf{F}_2 -dimension a .

Now the very simplicity of the G -module Q_B follows from Theorem 3.6.

The special case of $m = 4, q = 3$. We have $n = \#(B) = 40$ and $\dim_{\mathbf{F}_2}(Q_B) = 38$. According to the Atlas ([2], pp. 68-69), $G = \mathbf{L}_4(3)$ has two conjugacy classes of maximal subgroups of index 40. All other maximal subgroups have index greater than 40. Therefore all subgroups of G (except G itself) have index greater than $39 > 38$. This implies that each action of G on B is transitive. The permutation character (in notations of [2]) is (in both cases) $1 + \chi_4$, i.e., $\chi = \chi_4$. Since 40 is even, we need to consider the restriction of $\chi - 1$ to the set of 2-regular elements of G and this restriction coincides with the absolutely irreducible Brauer character ϕ_4 (in notations of [6], p. 165). In particular, the corresponding G -module Q_B is

absolutely simple. It follows from the Table on p. 165 of [6] that all absolutely irreducible representations of G in characteristic 2 have dimension which is *not* a strict divisor of 38. Combining this observation with the absence of subgroups in G of index less or equal than 38, we conclude, thanks to Theorem 3.6, that Q_B is very simple. This ends the proof of Theorem 3.7.

4. PROOF OF THEOREMS 2.1 AND 2.4

Recall that $\text{Gal}(f) \subset \text{Perm}(\mathfrak{R})$. In addition, it is known that the natural homomorphism $\text{Gal}(K) \rightarrow \text{Aut}_{\mathbf{F}_2}(J(C)_2)$ factors through the canonical surjection $\text{Gal}(K) \twoheadrightarrow \text{Gal}(K(\mathfrak{R})/K) = \text{Gal}(f)$ and the $\text{Gal}(f)$ -modules $J(C)_2$ and $Q_{\mathfrak{R}}$ are isomorphic (see, for instance, Th. 5.1 of [15]). In particular, if the $\text{Gal}(f)$ -module $Q_{\mathfrak{R}}$ is very simple then the $\text{Gal}(f)$ -modules $J(C)_2$ is also very simple and therefore is absolutely simple.

Lemma 4.1. *If the $\text{Gal}(f)$ -module $Q_{\mathfrak{R}}$ is very simple then either $\text{End}(J(C_f)) = \mathbf{Z}$ or $\text{char}(K) > 0$ and $J(C_f)$ is a supersingular abelian variety.*

Proof. This is Corollary 5.3 of [15]. □

It follows from Theorem 3.7 that under the assumptions of Theorem 2.1, the $\text{Gal}(f)$ -module $Q_{\mathfrak{R}}$ is very simple. Applying Lemma 4.1, we conclude that either $\text{End}(J(C_f)) = \mathbf{Z}$ or $\text{char}(K) > 0$ and $J(C_f)$ is a supersingular abelian variety.

If $n = 12$ and $\text{Gal}(f) \cong M_{12}$ then the $\text{Gal}(f)$ -module Q_B is also very simple ([15], Th. 7.12(ii)). Again we conclude that under the assumptions of Theorem 2.4 either $\text{End}(J(C_f)) = \mathbf{Z}$ or $\text{char}(K) > 0$ and $J(C_f)$ is a supersingular abelian variety ([15], Th. 7.13(ii)).

In order to finish the proof of Theorem 2.1 we need only to check that $J(C_f)$ is *not* supersingular if m is even. Similarly, in order to prove Theorem 2.4 we need only to check that if $(n, \text{Gal}(f)) = (12, M_{12})$ then $J(C_f)$ is *not* supersingular. Using Remarks 2.3 and 2.5, we may assume that either $\text{Gal}(f) = \mathbf{L}_m(q)$ or $(n, \text{Gal}(f)) = (12, M_{12})$ and in both cases K contains all 2-power roots of unity. Clearly, the desired assertions are immediate corollaries of the following statement.

Lemma 4.2. *Suppose an even positive integer n and a finite simple non-abelian group G enjoy one of the following two properties.*

- (i) *There exist an odd power prime q and an even integer $m \geq 4$ such that $n = (q^m - 1)/(q - 1)$ and $G \cong \mathbf{L}_m(q)$;*
- (ii) *$n = 12$ and $G \cong M_{12}$.*

Let us put $g = (n - 2)/2$. Suppose F is a field, whose characteristic is not 2. Suppose that F contains all 2-power roots of unity. Suppose that X is a g -dimensional abelian variety over F such that the image of $\text{Gal}(F)$ in $\text{Aut}(X_2)$ is isomorphic to G and the G -module X_2 is absolutely simple. Then X is not supersingular.

Proof of Lemma 4.2. Every nontrivial representation of G in characteristic 2 has dimension $> g$. Indeed, first assume that $G = \mathbf{L}_m(q)$. Then in the “generic” case” of $(m, q) \neq (4, 3)$ such a representation must have dimension $\geq 2g > g$, thanks to the already cited Th. 1.1 and Table III of [3]. If $(m, q) = (4, 3)$ then $n = 40$, $2g = 38$ and the smallest dimension is $26 > 19 = g$, according to the Tables in [6]. Second, if $G = M_{12}$ then this assertion follows from Th. 8.1 on p. 80 in [5]; see also the Tables in [6].

Proposition 4.3. *Suppose $G' \twoheadrightarrow G$ is a central extension of G . In addition, assume that either $G' = G$ or G' is a double cover of G , i.e., $\ker(G' \twoheadrightarrow G)$ is a central cyclic subgroup of order 2 in G' . Suppose V is a finite-dimensional \mathbf{Q}_2 -vector space and*

$$\rho : G' \rightarrow \mathrm{Aut}_{\mathbf{Q}_2}(V)$$

is an absolutely irreducible faithful representation of G' over \mathbf{Q}_2 . Then

$$\dim_{\mathbf{Q}_2}(V) \neq 2g.$$

Proof of Proposition 4.3. Clearly, ρ defines an absolutely irreducible projective representation of G in V over \mathbf{Q}_2 .

Assume first that $G = \mathbf{L}_m(q)$. Then in the “generic” case every absolutely irreducible nontrivial projective representation of G in characteristic 0 must have dimension $\geq 2g + 1 > 2g$ (see [3], Table II). If $(m, q) = (4, 3)$ then the Proposition follows from the Tables in [2].

Second, suppose $G = \mathbf{M}_{12}$. Then $n = 12, 2g = 10$. All faithful absolutely irreducible representations of \mathbf{M}_{12} in characteristic zero have dimension $\geq 11 > 10$ ([2], p. 33). This proves the Proposition in the case when $G' = G = \mathbf{M}_{12}$ and also when G' is a trivial double cover, i.e., is isomorphic to a product of \mathbf{M}_{12} and a cyclic group of order 2. If G' is a nontrivial double cover of \mathbf{M}_{12} then it has precisely two non-isomorphic 10-dimensional absolutely irreducible representations in characteristic 0 (up to an isomorphism) [4]. However, none of them is defined over \mathbf{Q}_2 . Indeed, each character of G' of degree 10 takes on a value, whose square is -2 ([4], Table 1 on p. 410; [2], p. 33). \square

Assume that X is supersingular. Our goal is to get a contradiction. We write $T_2(X)$ for the 2-adic Tate module of X and

$$\rho_{2,X} : \mathrm{Gal}(F) \rightarrow \mathrm{Aut}_{\mathbf{Z}_2}(T_2(X))$$

for the corresponding 2-adic representation. It is well-known that $T_2(X)$ is a free \mathbf{Z}_2 -module of rank $2\dim(X) = 2g$ and

$$X_2 = T_2(X)/2T_2(X)$$

(as Galois modules). Let us put

$$H = \rho_{2,X}(\mathrm{Gal}(F)) \subset \mathrm{Aut}_{\mathbf{Z}_2}(T_2(X)).$$

Clearly, the natural homomorphism

$$\bar{\rho}_{2,X} : \mathrm{Gal}(F) \rightarrow \mathrm{Aut}(X_2)$$

defining the Galois action on the points of order 2 is the composition of $\rho_{2,X}$ and the (surjective) reduction map modulo 2

$$\mathrm{Aut}_{\mathbf{Z}_2}(T_2(X)) \twoheadrightarrow \mathrm{Aut}(X_2).$$

This gives us a natural (continuous) surjection

$$\pi : H \twoheadrightarrow \bar{\rho}_{2,X}(\mathrm{Gal}(F)) \cong G,$$

whose kernel consists of elements of $1 + 2\mathrm{End}_{\mathbf{Z}_2}(T_2(X))$. We have assumed that the G -module X_2 is absolutely simple. This implies that the H -module X_2 is also absolutely simple. Here the structure of H -module is defined on X_2 via

$$H \subset \mathrm{Aut}_{\mathbf{Z}_2}(T_2(X)) \twoheadrightarrow \mathrm{Aut}(X_2).$$

The absolute simplicity of the H -module X_2 means that the natural homomorphism

$$\mathbf{F}_2[H] \rightarrow \text{End}_{\mathbf{F}_2}(X_2)$$

is surjective. By Nakayama's Lemma, this implies that the natural homomorphism

$$\mathbf{Z}_2[H] \rightarrow \text{End}_{\mathbf{Z}_2}(T_2(X))$$

is also surjective (see [10], p. 252).

Let $V_2(X) = T_2(X) \otimes_{\mathbf{Z}_2} \mathbf{Q}_2$ be the \mathbf{Q}_2 -Tate module of X . It is well-known that $V_2(X)$ is the $2g$ -dimensional \mathbf{Q}_2 -vector space and $T_2(X)$ is a \mathbf{Z}_2 -lattice in $V_2(X)$. Clearly, the $\mathbf{Q}_2[H]$ -module $V_2(X)$ is also absolutely simple.

The choice of polarization on X gives rise to a non-degenerate alternating bilinear form (Riemann form) [11]

$$e : V_2(X) \times V_2(X) \rightarrow \mathbf{Q}_2(1) \cong \mathbf{Q}_2.$$

Since F contains all 2-power roots of unity, e is $\text{Gal}(F)$ -invariant and therefore is H -invariant. This means that H is a subgroup of the corresponding symplectic group $\text{Sp}(V_2(X), e)$. We have

$$H \subset \text{Sp}(V_2(X), e) \cong \text{Sp}_{2g}(\mathbf{Q}_2) \subset \text{Sp}_{2g}(\bar{\mathbf{Q}}_2).$$

There exists a finite Galois extension L of K such that all endomorphisms of X are defined over L . We write $\text{End}^0(X)$ for the \mathbf{Q} -algebra $\text{End}(X) \otimes \mathbf{Q}$ of endomorphisms of X . Since X is supersingular,

$$\dim_{\mathbf{Q}} \text{End}^0(X) = (2\dim(X))^2 = (2g)^2.$$

Recall ([11]) that the natural map

$$\text{End}^0(X) \otimes_{\mathbf{Q}} \mathbf{Q}_2 \rightarrow \text{End}_{\mathbf{Q}_2} V_2(X)$$

is an embedding. Dimension arguments imply that

$$\text{End}^0(X) \otimes_{\mathbf{Q}} \mathbf{Q}_2 = \text{End}_{\mathbf{Q}_2} V_2(X).$$

Since all endomorphisms of X are defined over L , the image

$$\rho_{2,X}(\text{Gal}(L)) \subset \rho_{2,X}(\text{Gal}(F)) \subset \text{Aut}_{\mathbf{Z}_2}(T_2(X)) \subset \text{Aut}_{\mathbf{Q}_2}(V_2(X))$$

commutes with $\text{End}^0(X)$. This implies that $\rho_{2,X}(\text{Gal}(L))$ commutes with $\text{End}_{\mathbf{Q}_2} V_2(X)$ and therefore consists of scalars. Since

$$\rho_{2,X}(\text{Gal}(L)) \subset \rho_{2,X}(\text{Gal}(F)) \subset \text{Sp}(V_2(X), e),$$

$\rho_{2,X}(\text{Gal}(L))$ is a finite group. Since $\text{Gal}(L)$ is a subgroup of finite index in $\text{Gal}(F)$, the group $H = \rho_{2,X}(\text{Gal}(F))$ is also finite. In particular, the kernel of the reduction map modulo 2

$$\text{Aut}_{\mathbf{Z}_2} T_2(X) \supset H \twoheadrightarrow G \subset \text{Aut}(X_2)$$

consists of elements of finite order and, thanks to the Minkowski-Serre Lemma, $Z := \ker(H \rightarrow G)$ has exponent 1 or 2. In particular, Z is commutative. We have

$$Z \subset H \subset \text{Sp}(V_2(X), e) \cong \text{Sp}_{2g}(\mathbf{Q}_2) \subset \text{Sp}_{2g}(\bar{\mathbf{Q}}_2).$$

Since Z consists of semisimple elements and rank of Sp_{2g} is g , the group Z is isomorphic ("conjugate") to a multiplicative subgroup of $(\mathbf{Q}_2^*)^g$. Since the exponent of Z is either 1 or 2, the group Z is isomorphic to a multiplicative subgroup of $\{1, -1\}^g$. Hence Z is an \mathbf{F}_2 -vector space of dimension $d \leq g$. This implies that the adjoint action

$$H \rightarrow H/Z = G \rightarrow \text{Aut}(Z) \cong \text{GL}_d(\mathbf{F}_2)$$

is trivial, since every nontrivial representation of G in characteristic 2 must have dimension strictly greater than $g \geq d$. This means that Z lies in the center of H . Since the $\mathbf{Q}_2[H]$ -module $V_2(X)$ is faithful and absolutely simple, Z consists of scalars. This implies that either $Z = \{1\}$ or $Z = \{\pm 1\}$. In other words, either $H \cong G$ or $H \twoheadrightarrow G$ is a double cover. In both cases $V_2(X)$ is an absolutely irreducible representation of H of dimension $2g$ over \mathbf{Q}_2 . But by Proposition 4.3 applied to $G' = H$ and $V = V_2(X)$,

$$\dim_{\mathbf{Q}_2}(V_2(X)) \neq 2g.$$

This gives us the desired contradiction. This ends the proof of Lemma 4.2 and therefore of Theorems 2.1 and 2.4. \square

Example 4.4. Suppose p is an odd prime, $q > 1$ is a power of p , $m > 2$ is an even integer. Let us put $n = (q^m - 1)/(q - 1)$. Suppose k is an algebraically closed field of characteristic p and $K = k(z)$ is the field of rational functions. The Galois group of $x^m + zx + 1$ over K is $\mathbf{L}_m(q)$ and the Galois group of $x^m + x + z$ over K is $\mathrm{PGL}_m(\mathbf{F}_q)$ ([1], p. 1643). Therefore the jacobians of the hyperelliptic curves $y^2 = x^m + zx + 1$ and $y^2 = x^m + x + z$ have no nontrivial endomorphisms over an algebraic closure of K .

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